EQUILIBRIUM PRICES OF ASSETS AND THE THEORY OF RISK-BEARING

THE EXPONENTIAL UTILITY FUNCTION CASE

by

A. Anastasopoulos
Department of Economics
Sir George Williams University
Montreal, Quebec
Canada

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The theory of portfolio selection is well established by now. Furthermore, the merits of exponential functions in representing the attitude towards risk of investors have been explored by J. Pratt [5]. What this note attempts to add is the implications of exponential utility functions for (i) the composition of portfolios in a competitive equilibrium and (ii) the equilibrium prices of assets.

The main results of this note are: (i) the amount of each asset which an investor holds in equilibrium depends only on his measure of risk aversion, [1, Ch. 3 and 5] (as indicated by the parameter of the exponent relative to the measures of risk aversion of all other investors; (2) the equilibrium price of an asset is the weighted sum of the cumulants [6, p.68] of its distribution of returns. The weights can be considered as the prices which the community assigns to each cumulant, and they depend only on the number and the measure of risk aversion of those who trade in the asset market -- Thus, we have a well defined price of risk-bearing.

The Price of Risk-Bearing

In a competitive market there are n individuals and m risky assets. The total existing quantity of the jth asset, $j = 1, 2, \ldots m$ is one unit which sells for $\$p_j$ and which will yield a random gross return x_j. The return of the ith asset is assumed to be independent of the returns of all other assets and its distribution, $F_j(x_j)$, objectively known to all individuals. Furthermore, we assume that the expected net returns of the jth

asset are positive, i.e. $E[x_j] - p_j > 0$ where E stands for "expectation."

The ith individual has wealth W_i which he plans to invest in assets. Let $(\alpha_{i1}, i_2, \ldots i_m)$ be a portfolio of assets of the ith individual. His optimum portfolio $(\alpha^*_{i1}, \alpha^*_{i2}, \ldots \alpha^*_{im})$ is the one which maximizes.

$$(1) \quad E[-\exp\{-\lambda_i \sum_j \alpha_{ij} x_j\}] = -\prod_{j=1}^{m} E[\exp\{-\lambda_i \alpha_{ij} x_j\}]$$

$$\text{subject to } \sum_{j=1}^{m} \alpha_{ij} p_j = W_i \quad \text{and} \quad \alpha_{ij} \geq 0 \text{ for all } j,$$

where $-\exp(-\lambda_i W_i)$ is the utility of wealth of the ith individual, and $\lambda_i > 0$ is the measure of his risk aversion [5, p.130].

Multiply (1) by -1 and take its logarithm to obtain

$$(2) \quad \ln \prod_{j=1}^{m} E[\exp\{-\lambda_i \alpha_{ij} x_j\}] = \sum_{j=1}^{m} \ln E[\exp\{-\lambda_i \alpha_{ij} x_j\}]$$

Suppose that $\lambda_i \alpha_{ij}$ is a small number. Then, we can expand the expression $E[\exp\{-\lambda_i \alpha_{ij} x_j\}]$ in (2) in Maclaurin series around zero. We assume that the series converge. Furthermore, if we expand the logarithms under the summation sign of (2) we obtain:

$$(3) \quad \sum_{j=1}^{m} \ln\{1 + \sum_{s=1}^{\infty} \frac{(-1)}{s!} (\lambda_i \alpha_{ij})^s \mu'_{js}\} = \sum_{j=1}^{m} \sum_{s=1}^{\infty} \frac{(-1)^s}{s!} (\lambda_i \alpha_{ij})^s k_{js}$$

where k_{js} is the sth cumulant of $F_j(x_i)$ and $\alpha^s_{ij} k_{js}$ is the sth cumulant of the distribution of $\alpha_{ij} x_j$.

To maximize (1) subject to the wealth constraint, it is enough to minimize (3) subject to the same constraint. The term-by-term differentiation of (3) with respect to α_{ij} is permissible in this case [3, p.161]. The first order conditions are:

$$(4)\quad \sum_{s=1}^{\infty} \frac{(-1)^s}{(s-1)!} \lambda_i^s \alpha_{ij}^{s-1} K_{js} - \delta_i p_j \leq 0 \quad , \quad j = 1, 2, \ldots m$$

$$\sum_{j=1}^{m} \alpha_{ij} p_j = W_i$$

where δ_i is the Lagranzean multiplier--It is easy to show that the second order conditions for a minimum are satisfied.

Suppose that the mth asset is money, that money is barren (i.e. $p_m \equiv k_{ml} \equiv \mu_{ml}$), and that α_{im} can be positive or negative (i.e. borrowing is allowed). Then the mth first order condition reduces to $-\lambda = \delta_i$. Furthermore, it can be easily shown that $\alpha_{ij} > 0$ if and only if $\mu_{jl} > \vec{p_j}$ for $j = 1, 2, \ldots m-1$, so that all relations in (4) are equalities. Substitute $(-\lambda_i)$ for δ_i in (4) and consider these conditions for the ith and the Kth individuals:

$$(5)\quad p_j = \sum_{s=1}^{\infty} \frac{(-1)^{s-1}}{(s-1)!} (\lambda_i \alpha_{ij}^*)^{s-1} k_{js} = \sum_{s=1}^{\infty} \frac{(-1)^{s-1}}{(s-1)!} (\lambda_K \alpha_{Kj}^*)^{s-1} K_{js}$$

$$j = 1, 2, \ldots m-1$$

From (5) we derive the following theorem:

Theorem - $\lambda_K \alpha_{Kj}^* = \lambda_i \alpha_{ij}^*$, $j = 1, 2, \ldots, m$ and $K, i \in \{1, 2, \ldots, n\}$

In words, given any two individuals, the ratio of their optimum holdings of an asset $(\frac{\alpha_{kj}}{\alpha_{ij}})$ is equal to the inverse of the ratio of their measures of risk aversion $(\frac{\alpha_{kj}}{\alpha_k})$. The proof is given in the appendix.

If we divide the expression in the theorem by λ_i, sum over i, and set this sum equal to the total quantity of the ith asset (which is one unit) we obtain:

Corollary 1

In equilibrium , $\alpha^*_{kj} = \dfrac{1}{\lambda_k \sum_{i=1}^{n} \frac{1}{\lambda_i}}$, $k = 1, 2, \ldots n$

$j \quad 1, 2, \ldots n$

i.e. the optimum amount of a risky asset which an individual holds in equilibrium depends only on his measure of risk aversion relative to the attitude towards risk of all the other members of the community.

If we use Corollary 1 in (5) we obtain the price of an asset as the weighted sum of its cumulants:

Corollary 2

$$p_j = \sum_{s=1}^{n} \pi_{sn} k_{js} \qquad , j = 1, 2, \ldots$$

where $$\pi_{sn} \equiv \frac{(-1)^{s-1}}{(s-1)!} \left(\sum_{i=1}^{n} \frac{1}{\lambda_i} \right)^{s-1}$$

The weight π_{sn} can be considered as the price which the community assigns to the sth cumulant of any distribution. These prices depend on the attitude towards risk and the number of individuals trading in the market.

K. Borch has suggested that if the prices of assets are to imply a price for the service of risk-bearing in a meaningful way,

cumulants should occur as separate commodities, each with its own market price [2]. The discussion above shows that each individual chooses his optimum portfolio as if he attaches subjectively prices to the cumulants of distributions. However, only in equilibrium can we talk about "market prices" of cumulants.

Finally, it can be easily seen that if λ_i's are finite, $\lim_{n\to\infty} \pi_{sn} = 0$ for $s > 1$. Furthermore, one can easily establish the convergence of the double limit involved in $p_j = \sum_{s=1}^{n} \pi_{sn} K_{js}$. Therefore, $\lim_{n\to\infty} p_j = u_j \quad , \qquad j = 1, 2, \ldots$

i.e. as the number of individuals increases the equilibrium price of an asset approaches, in the limit, its expected returns. This holds either when the existing quantity of the risky asset remains constant or increases with the number of individuals.

APPENDIX - Proof of the Theorem

To prove the theorem we must first show that (2) and, therefore, (3) are monotonic in the neighborhood of $(\alpha^*_{i1}, \alpha^*_{i2}, \ldots, \alpha^*_{im-1})$.

To this end, it is enough to show that $\ln E[\exp\{-\lambda_i \alpha_{ij} x_j\}]$ is monotonic around α^*_{ij} for any j, $j = 1, 2, \ldots m$. In the proof we will omit subscripts in order to simplify the notation.

Given any two amounts of the same asset α^1, α^2, consider the expression:

$$E[\exp\{-\tfrac{1}{2}\lambda(\alpha^1+\alpha^2)x\}] < E[\{\exp(-\tfrac{1}{2}\lambda\alpha^1 x)^2\}]E[\{\exp(-\tfrac{1}{2}\lambda\alpha^2 x)^2\}],$$

by Hölder's inequality. By taking logarithms of both sides we obtain:

$$\ln E[\exp\{-\tfrac{1}{2}\lambda(\alpha^1+\alpha^2)x\}] < \tfrac{1}{2}\ln E[\exp(-\lambda\alpha^1 x)] + \tfrac{1}{2}\ln E[\exp(-\lambda\alpha^2 x)],$$

i.e. $\ln E[\exp\{-\lambda\alpha x\}]$ is a strictly convex function of α. Furthermore, from the first order conditions (4) and $-\lambda_i = \delta_i$ we have

$\frac{d}{d\alpha}\ln E[\exp\{-\lambda\alpha x\}] = -\lambda p_j < 0$ which shows that $\ln E[\exp\{-\lambda\alpha x\}]$ is monotonically decreasing in the neighborhood of α^*.

The remaining of the proof follows easily from what it was said above and the uniqueness of the expansion in (3).

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